PRACTICAL EXTENSIONS OF CONNECTED OPERATORS

P. SALEMBIER and A. OLIVERAS *
Dept. of Signal Theory and Communications
Universitat Politècnica de Catalunya
Campus Nord - Módulo D5
C/ Gran Capità, 08034 Barcelona, Spain
Tel: (343) 401 74 04, Fax: (343) 401 64 47
E-mail: philippe@gps.tsc.upc.es

Abstract. This paper deals with the notion of *connected operators*. These operators are becoming popular in image processing because they have the fundamental property of simplifying the signal while preserving the contour information. In this paper, we discuss some practical approaches for the extension and the generalization of these operators. We focus on two important issues: the simplification criterion and the connectivity. We present in particular complexity- and motion-oriented connected operators. Moreover, we discuss the creation connectivities that are either more or less strict than the usual ones.

Key words: Mathematical morphology, Connected operators, Nonlinear filtering, Complexity criterion, Motion criterion, Connectivity, Watersheds, Opening, Closing.

1. Introduction

The first *connected operators* reported in the literature are known as *opening by reconstruction* [4]. This original idea led to geodesic operators on sets [5], to markers for numerical functions [7], to multi-resolution decomposition with filters by reconstruction [10], to the concept of dynamics [2], to area opening [14], to the *flat zones* approach to segmentation [1]. Moreover, an intensive work has been done on the efficient implementation of these transformations [15]. These *connected operators* involved not only openings but also closings, alternated filters or even alternating sequential filters. They are becoming very popular because, on experimental bases, they have been claimed to simplify the image while preserving contours. This rather surprising property makes them very attractive for a very large number of applications such as noise cancellation, segmentation, pattern recognition, etc.

The study reported in [13, 11] revealed that filters by reconstruction belong to a larger class of transformation called *connected operators* that have the fundamental property of interacting with the signal by means of connected components (in the case of binary images) or of *flat zones* (in the case of gray level images). This viewpoint opens the door to various generalization of connected filters. Two lines of generalizations are reported in [8]. The first one deals with the simplification criterion and a new criterion called *complexity* was proposed. The second generalization deals with the connectivity. This notion is closely related to the definition

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of elementary objects in the scene. The connectivity can be modified to eliminate some drawbacks of connected filters such as the so-called *leakage*. In [8], we discussed notions that are either more or less strict than the "usual" connectivity used in digital image processing. The objective of this paper is to make a summary of these two lines of generalization and to further investigate the approach. We will introduce in particular a new simplification criterion which is motion-oriented, and we will propose a practical solution to handle a theoretical problem related to "strict" connectivity.

The organization of this paper is as follows: section 2 is devoted to the notion of binary and gray level connected operators. Section 3 deals with the simplification criterion, whereas section 4 focuses on the notion of connectivity. Finally, section 5 presents the conclusions and discusses possible extensions of this work.

2. Binary and Gray-level Connected Operators

The original idea of *binary connected operators* relies on the separation of an analysis step and of a decision step as illustrated in Fig. 1. The first one assesses a characteristic of a binary connected component following a given criterion, whereas the second one states whether or not a connected component has to be preserved.

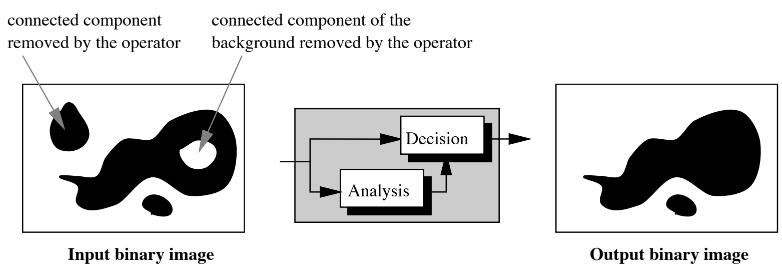


Fig. 1. Example of binary connected operator

In [13, 11], the concept of binary connected operators is formally defined as follows: first, a connectivity has to be defined. In the discrete case, this step generally reduces to the definition of a local neighborhood system describing the connections between adjacent pixels. The classical choices involve 4-, 6- or 8-connectivity, however, in section 4, we will come back to this notion. Once the connectivity has been selected, the notion of connected operators can be defined as follows:

Definition 2.1 (Binary connected operators) *A binary operator ψ is connected when for any binary image X, the symmetrical difference $X \setminus \psi(X)$ is exclusively composed of connected components of X or of its complement X^c.*

The extension of connected operators for gray level functions relies on the concept of partition [13, 11]. Let us recall that a partition of the space E is a set of connected components $\{A_i\}$ which are disjoint and the union of which is the entire space. Each A_i is called a partition class. Moreover, a partition $\{A_i\}$ is said to be *finer* than another partition $\{B_i\}$ if any pair of points belonging to the same class A_i also belongs to a unique partition class B_j. Consider now a binary image and define its *associated partition* as the partition made of the connected components of the binary sets and of their complements. The definition of connected operators can be expressed with associated partitions:

Theorem 2.2 (Binary connected operators via partition) *A binary operator ψ is connected if and only if, for any binary image X, the associated partition of $\psi(X)$ is less fine than the associated partition of X.*

The concept of gray level connected operators can be introduced if we define a partition associated to a function. To this end, the use of *flat zones* was proposed in [13, 11]. The set of flat zones of a gray level function f is the set of the largest connected components of the space where f is constant (note that a flat zone can be reduced to a single point). It can be demonstrated [13] that the set of flat zones of a function constitutes a partition of the space. This partition is called the *partition of flat zones* and leads to the following definition:

Definition 2.3 (Gray level connected operators) *An operator Ψ acting on gray level functions is connected if, for any function f, the partition of flat zones of $\Psi(f)$ is less fine than the partition of flat zones of f.*

There are several ways of creating gray level connected operators. A simple one, relying on *threshold decomposition* and *stacking*, is illustrated in Fig. 2. The *threshold decomposition* generates one binary image X_λ for each possible gray level value λ. The binary image is decomposed into a set of connected components that are processed by the binary connected operator ψ. Finally, the stacking consists in reconstructing a gray level image $g = \Psi(f)$ from the set of binary images $\psi(X_\lambda)$ [6, 3, 12]:

$$g = \Psi(f) = \bigvee_{\lambda} \Big(\bigcap_{\mu < \lambda} \psi(X_\mu) \Big) \tag{1}$$

Note that if the binary connected operator ψ is increasing, the stacking can be simplified:

$$g = \Psi(f) = \bigvee_{\lambda} (\psi(X_\lambda)) \tag{2}$$

Following this procedure, it can be shown [13, 11] that the resulting gray level operator Ψ is a connected operator because the partition of flat zones of f is always finer than the partition of flat zones of $\Psi(f)$.

This way of creating connected operators opens the door to several generalization. In this paper, we will focus on two points: first, the *analysis* step of Fig. 1. As can be seen, by modifying the criterion that is assessed in this block, a large set of binary as well as gray level connected operators can be created. Second, the *connectivity definition* that is defined after the thresholding operation in Fig. 2. This processing step defines the elementary image objects on which the decision is going to interact. A modification of the definition of the connected components after thresholding leads to a different notion of elementary objects.

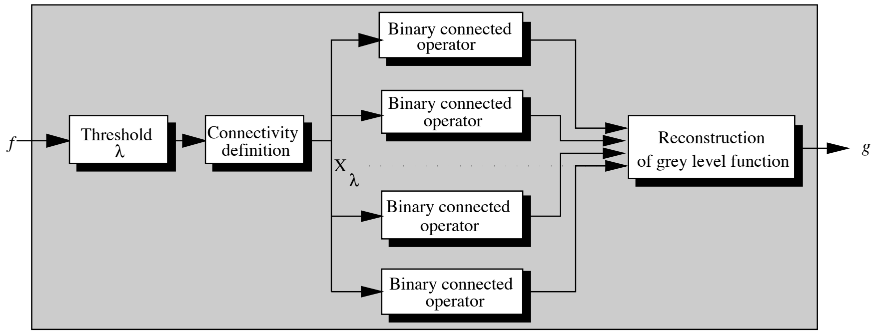


Fig. 2. Example of construction of gray level connected operator from a binary connected operator

3. Filtering criterion

3.1. Classical Criteria

As examples, let us briefly recall the classical criteria used for the *opening by reconstruction*, the *area opening* and the *h-max* operator. The first two operators can deal with binary (scheme of Fig. 1) as well as gray level images (scheme of Fig. 2) whereas the last one is devoted to gray level images only.

- *Opening by reconstruction* [4]: This filter preserves all connected components that are not totally removed by a binary erosion by a structuring element of size h. This opening has a size-oriented simplification effect: in the case of gray level images, it removes the bright components that are smaller than the structuring element. By duality, a closing by reconstruction can be defined. Its simplification effect is similar to that of the opening but on dark components.

- *Gray level area opening* [14]: This filter is similar to the previous one except that it preserves the connected components that have a number of pixels larger than a limit h. It is also an opening which has a size-oriented simplification effect, but the notion of size is different from the one used in the *opening by reconstruction*. By duality an *area closing* can be defined.

- $h - max$ operator: The criterion here is to preserve a connected component of the binary image X_μ if and only if this connected component hits a connected component of the binary image $X_{\mu+h}$. This is an example where the criterion involves two binary images obtained at two different threshold values. The simplification effect of this operator is contrast-oriented in the sense that it eliminates image components with a contrast lower than h. Note that, the $h - max$ is an operator and not a morphological filter because it is not idempotent. By duality, the $h - min$ operator can be defined.

3.2. Complexity Criterion

In [8], a connected operator dealing with the complexity of objects is proposed. The idea is to define a binary connected operator that removes complex binary connected components. To this end, the simplification criteria relies on the ratio between the perimeter $\mathcal{P}$ and the area $\mathcal{A}$. Intuitively, it can be seen that if a connected component has a small area but a very long perimeter, it corresponds to a complex object.

Definition 3.1 (Complexity criterion)

$$\mathcal{C} = \mathcal{P}/\mathcal{A} \tag{3}$$

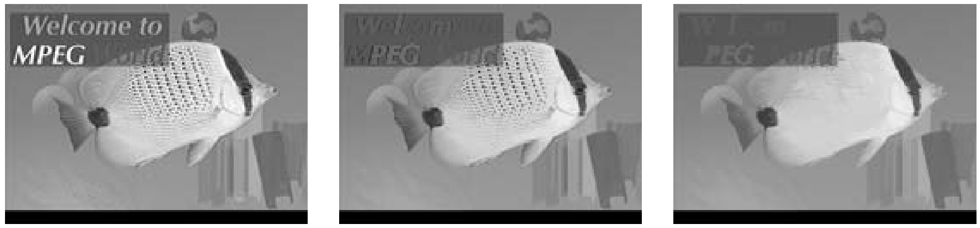
The complexity criterion is not an increasing criterion because if the set X is included in the set Y, there is *a priori* no relation between their complexity. The reconstruction of the gray level function can therefore be achieved by the formula of Eq. 1. However, as discussed in [8], this reconstruction process severely decreases the contrast of the image. In practice, the reconstruction defined by Eq. 2 leads to more useful results and is assumed to be used in the sequel.

The complexity operator is idempotent, anti-extensive but non increasing. It is therefore not a morphological filters in the strict sense. In practice, this operator removes complex and bright objects from the original image. A dual operator dealing with the complexity of dark objects can be easily defined. An example of processing can be seen in Fig. 3. The original image is composed of various objects with different complexity. In particular the text and the texture of the fish can be considered as being complex by comparison with the shape of the fish and the books on the lower right corner. Fig. 3.B shows the output of the complexity operator. On this result, a dual complexity is applied (Fig. 3.C). This can be considered as an alternated operator. As illustrated on this example, the complexity operators efficiently remove complex image components (text and texture of fish) while preserving the contours of the objects that have not been eliminated. In both cases, the filters have removed objects of complexity higher that 1 in the sense of Eq. 3. Note that the simplification effect is not size-oriented, because the filters have removed large objects (the "MPEG" word) as well as small objects (the texture of the fish). The simplification is not contrast-oriented as can be seen by the difference in contrast between "Welcome to" and "MPEG" which have been jointly removed.

3.3. Motion Criterion

In this section, a new connected operator allowing to deal with the *motion* information in an image sequence is introduced. The idea is to define a binary connected operator removing binary connected components that do not undergo a given motion and to extend this operator for gray level images by the scheme of Fig. 2.

Consider two consecutive frames and assume a translation as motion model (see Fig. 4). Suppose, for instance, that we would like to eliminate all connected components of the current (binary) frame (at time T) that do not undergo a given translation (V_x, V_y). A simple solution consists in looking in the next frame (at time $T+1$) at the location defined by the translation if the same connected component is



A) Original Image | B) Result of the complexity operator | C) Result of the dual complexity operator

Fig. 3. Example of processing with the complexity connected operator

present. If this is the case, the connected component of the current frame is retained otherwise it is removed.

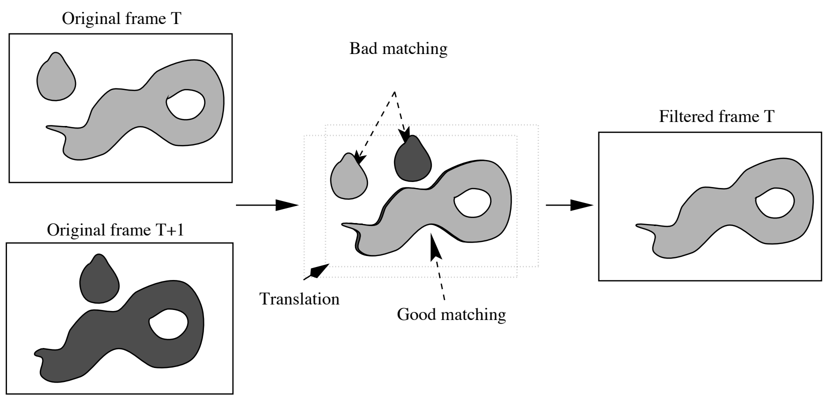


Fig. 4. Binary motion connected operator

In practice, the matching between two connected components is not perfect and a given tolerance of mismatch $\mathcal{M}$ (measured in % of matching pixels) has to be accepted. The gray level operator is generated from the binary operator by using the scheme of Fig. 2. Note that here also, the motion criterion is not increasing. The gray level operator has the ability to remove bright objects from the scene that do not undergo a given motion.

Several filtering results can be seen in Fig. 5. The large bright boat on the left side of the picture moves following a translation $(V_x, V_y) = (19, 0)$. Fig. 4 allows the estimation of the influence of the motion (V_x, V_y) and mismatch $\mathcal{M}$ parameters. As can be seen in the central column of Fig. 5, the filter has preserved the large boat and has removed most of the remaining bright image components. In this figure one can also see that a proper mismatch is $\overline{\mathcal{M}} \approx 88\%$.

Fig. 6 illustrates the results obtained with the connected operator followed by its

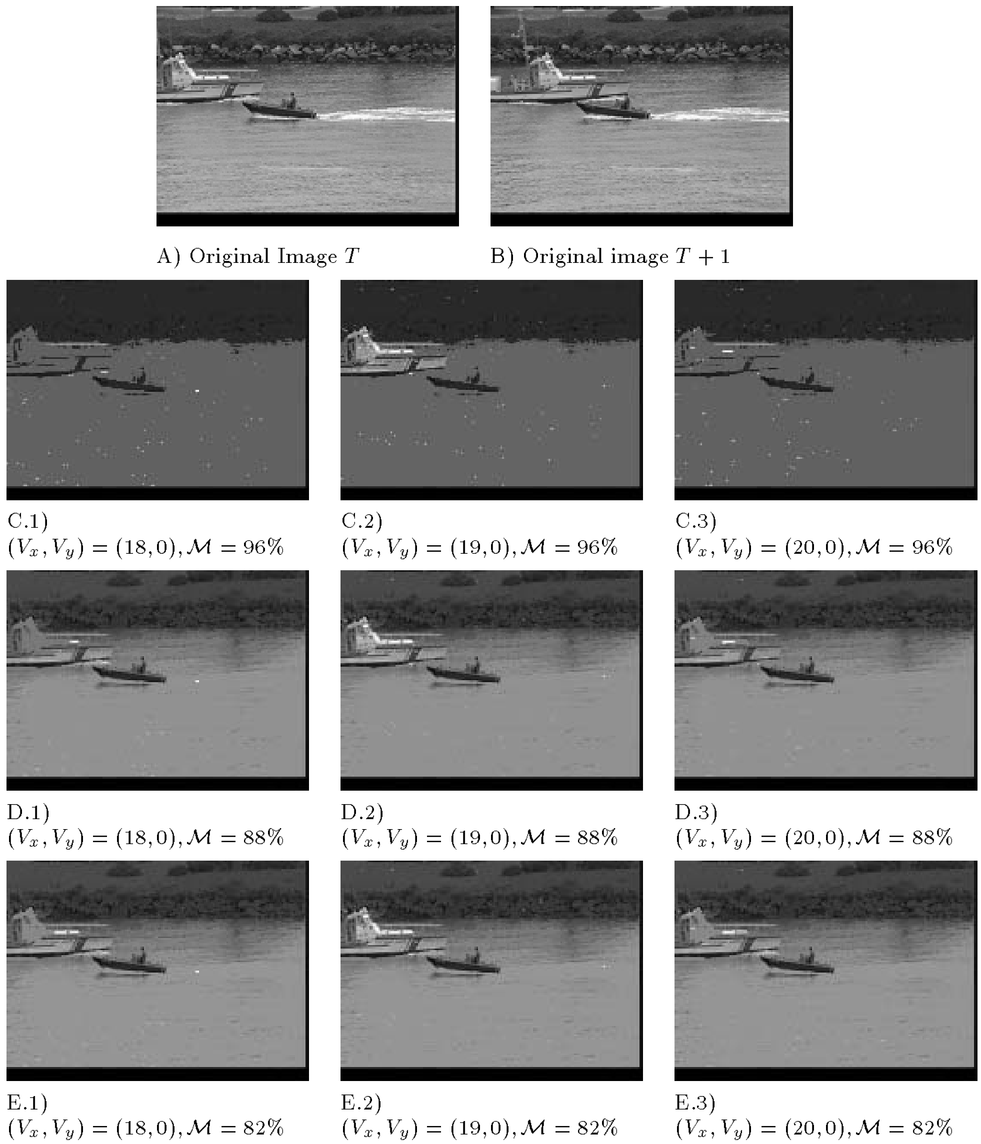
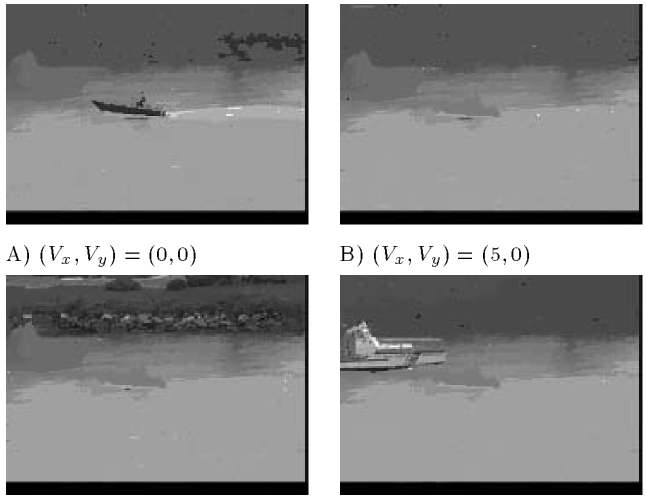


Fig. 5. Example of processing with the motion connected operator

dual for four values of the translation. Now the simplification deals with bright as well as dark objects. As can be seen, the connected operator allows to extract the small boat (A:$(V_x, V_y) = (0,0)$), the background (C:$(V_x, V_y) = (10,0)$) and the large boat (D:$(V_x, V_y) = (19,0)$). Moreover, it does not extract any particular objects if no objects follows a given motion (B:$(V_x, V_y) = (5,0)$).



A) $(V_x, V_y) = (0,0)$ B) $(V_x, V_y) = (5,0)$

C) $(V_x, V_y) = (10,0)$ D) $(V_x, V_y) = (19,0)$

Fig. 6. Example of processing with the motion connected operator ($\mathcal{M} = 88\%$)

4. The Connectivity

In discrete space, the notion of connectivity usually relies on the definition of a local neighborhood system that defines the set of pixels that are connected to a given point. In practice, 4-, 6- and 8-connectivity are the most popular choices. In the examples of the previous section, a 4-connectivity was used. The objective of this section is to discuss some possible extensions of the connectivity notion and its influence on the resulting set of connected operators.

4.1. Classical Connectivity

The notion of connectivity has been introduced in morphology [12] starting from the following definition:

Definition 4.1 (Connectivity class) *A* connectivity class $\mathcal{C}$ *is defined on the subsets of a set* E *when:*

1. $\emptyset \in \mathcal{C}$ *and* $\forall\ x \in E$, $\{x\} \in \mathcal{C}$
2. *For each family* $\{\mathcal{C}_i\}$ *of* $\mathcal{C}$, $\bigcap \mathcal{C}_i \neq \emptyset \Rightarrow \bigcup \mathcal{C}_i \in \mathcal{C}$

It was shown in [12] that this definition is equivalent to the definition of a family of *connected pointwise openings* $\{\gamma_x, x \in E\}$ associated to each point of E:

Theorem 4.2 (Connectivity characterized by openings) *The definition of a connectivity class* $\mathcal{C}$ *is equivalent to the definition of a family of openings* $\{\gamma_x, x \in E\}$ *such that:*

1. $\forall\ x \in E$, $\gamma_x(\{x\}) = \{x\}$

2. $\forall\ x,\ y \in E$ *and* $X \subseteq E$, $\gamma_x(X)$ *and* $\gamma_y(X)$ *are either equal or disjoint.*
3. $\forall\ x \in E$ *and* $X \subseteq E$, $x \notin X \Rightarrow \gamma_x(X) = \emptyset$

Intuitively, the opening $\gamma_x(X)$ is the connected component of X that contains x. Based on this definition of the connectivity, a generalization was proposed in [12]. It relies on the definition of a new connected pointwise opening:

$$\nu_x(X) = \gamma_x(\delta(X)) \bigcap X \text{ , if } x \in X \qquad \text{and} \qquad \nu_x(X) = \emptyset \text{ , if } x \notin X \qquad (4)$$

where δ is an extensive dilation. It can be demonstrated that this new function is indeed a connected pointwise opening and therefore defines a new connectivity. This connectivity is less "strict" than the usual ones in the sense that it considers that two objects that are close to each other (that is they touch each other if they are dilated by δ) belong to the same connected component. This generalization can lead to interesting new connected filters, however, in order to have a flexible tool one would like also to define connectivities that are more "strict" that the usual ones, that is they should split what is usually considered as one connected component. In [8] such a tool was proposed. However, it was shown that the resulting notion is not a real connectivity. The purpose of the following section is to discuss this issue of "strict" connectivity.

4.2. "Strict" Connectivities and Pseudo-connectivities

The intuitive idea of "strict" connectivity relies on the segmentation of the binary connected components. Indeed, the objective is to split the connected components into a set of elementary shapes that are going to be processed separately. The connected operator will take individual decision on each elementary shape. Ideally, the shapes should correspond to our perception of the main parts of the object.

To our knowledge, two attempts have been reported in the literature to define "strict" connectivities.

- *Segmentation by openings* [9]: Given a family of *connected pointwise openings*, γ_x, and an opening γ, a new family of *connected pointwise opening*, σ_x can be created by the following rule:

  $$\sigma_x(X) = \gamma_x\gamma(X), \text{ if } x \in \gamma(X) \qquad \text{and} \qquad \sigma_x(X) = \{x\}, \text{ if } x \in X \setminus \gamma(X) \quad (5)$$

  and as usual $\sigma_x(X) = \emptyset$, if $x \notin X$. It is shown in [9] that σ_x is actually a *connected pointwise opening* and therefore defines a connectivity. Intuitively, this connectivity considers that the connected components of a binary sets are made of the connected components of its opening by γ and the points that are removed by the opening are considered as isolated points, that are connected components of size one.

  Even if this solution is theoretically sound, in practice it turns out that this way of segmenting the connected components leads to a loss of one of the main features of connected operators. In practice, connected operators are used because they can simplify while preserving the shape information of the remaining image components. Suppose we use an area opening of size larger than one with the connectivity defined by the *connected pointwise opening* of Eq. 5. The filter will eliminate all the isolated points (area equal to one) and all the small connected components resulting from the opening. The shape information of the remaining components will not be preserved because most of the time, this shape information relies on the set of isolated points.

– *Segmentation by watershed* [8]: The idea is to rely on classical binary segmentation tools (see [7] and the references herein). One of the simplest approaches consists in computing the distance function $Dist_X$ on the binary set X and in computing the *watershed* of $-Dist_X$. The *watershed* transform associates to each minima of $-Dist_X$ a region called a *catchment basin*. Note that the minima of $-Dist_X$ are the maxima of the distance function, in other words, they correspond to the *ultimate erosions* of the set.

  If this segmentation driven by the ultimate erosion creates too many connected components, the number of connected components can be defined by the number of connected components in the classical sense of an erosion of size l of X. This can be implemented via the segmentation of a thresholded version of the distance function:

$$\mathcal{D}_{X,l} = -(Dist_X \bigwedge l) \tag{6}$$

  The parameter l of $\mathcal{D}_{X,l}$ allows to go progressively from the classical connectivity when $l = 0$ to the extreme case where the number of connected components are defined by the number of ultimate erosions when $l = \infty$. Note that one can easily integrate within the same framework the loose connectivity described by the *connected pointwise opening* ν_x of Eq. 4 by taking into account the distance function of the background (see [8] for more details).

  Let us define $\mathcal{CB}^l_x(X)$ the transformation that assigns to x the *catchment basin* of the function $\mathcal{D}_{X,l}$ that contains x. Consider now the operator:

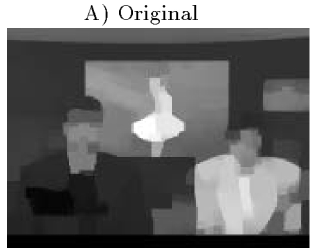
$$\mathcal{CC}^l_x(X) = \mathcal{CB}^l_x(X) \bigcap X, \text{ if } x \in X \qquad \text{and} \qquad \mathcal{CC}^l_x(X) = \emptyset, \text{ if } x \notin X \tag{7}$$

  This transformation reduces to the classical *connected pointwise opening* γ_x when $l = 0$. For $l > 0$, it only creates a *pseudo-connectivity*. Indeed, in that case, all conditions of theorem 4.2 are met except one: $\mathcal{CC}^l_x$ is not increasing and therefore not an opening. This is a drawback, but, using the watershed as segmentation tool, our main concern is to segment the component of X in a small number of regions and to keep as much as possible the contour information of X, because it is one of the main attractive properties of connected operators. Moreover, in practice for small values of l, this theoretical problem does not prevent the creation of useful operators.

Fig. 7 illustrates several examples of area open-close [14] with several notions of connectivity. The classical area open-close (4-connectivity) can be seen in Fig. 7.B. This example illustrates a typical problem of connected operators called *leakage*.

A) Original

B) Classical 4-connectivity



C) Connectivity obtained by segmentation by opening (opening by a square of 3*3)

D) Connectivity obtained by segmentation by watershed ($l = 1$)

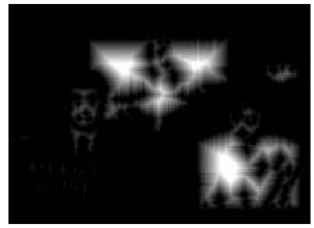
Fig. 7. Example of area filtering (open-close) with "strict" connectivity, area parameter $\lambda = 100$

Small objects like the letters of the "MPEG4" word should have been removed. This is however not the case for the "E" and the "G" because there is a thin connection between these letters and the shirt of the man. Using the classical connectivity, the operator processes the shirt and the "E" and "G" letters as a single object and the connected operator reconstructs "too much". Fig. 7.C shows the result obtained by the "strict" connectivity of Eq. 5. As can be seen, the contour preservation property is lost. Finally, Fig. 7.D gives the result obtained by the "strict" pseudo-connectivity of Eq. 7. The leakage problem has disappeared and the contour preservation property is not lost. In this example, thin connections between components are broken and the final result corresponds more to a "natural" size-oriented simplification.

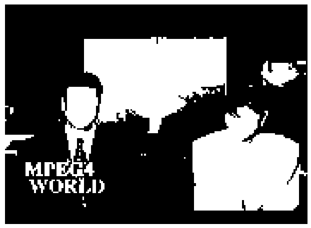
4.3. Robust Pseudo-connectivity

As discussed in [8], the fact that $\mathcal{CC}_x^l$ is not increasing leads to a lack of robustness in the definition of the connectivity. In practice, this phenomenon is a problem for large values of l. This drawback can be seen in Fig. 9.A. This example correspond to the same filter as the one of Fig. 7.D but with $l = \infty$. In other words, all connected components are segmented and the number of regions is defined by the ultimate erosions. The lack of robustness leads to the apparition of false contours. In order to improve the robustness of pseudo-connectivity one can either modify the segmentation strategy or the signal to segment. The first option seems difficult to achieve without loosing the contour preservation property of the operator. Therefore, we will focus on the second option.

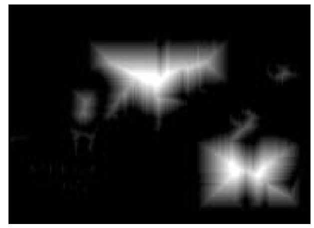
A) Binary components of X



B) Distance $Dist_X$



C) Binary components of a closing by reconstruction $\varphi^{rec}(X)$



D) Distance $Dist_{\varphi^{rec}(X)}$

Fig. 8. Comparison between the distance function of X and of its closing by reconstruction $\varphi^{rec}(X)$

A careful analysis of the segmentation results shows that a large part of the lack of robustness of the segmentation comes from small holes in the connected components. This is illustrated in Fig. 8. Fig. 8.A shows an thresholded version of an image and Fig. 8.B presents the corresponding distance function. As can be seen, the distance function is rather complex and, because of the presence of small holes within the connected components, the distance function possesses a large number of regional maxima. As a result, a large number of connected components will be created by the segmentation. This problem can be partially overcome if the distance function is computed not on the original binary image X but on the result of a closing by reconstruction $\varphi^{rec}(X)$ (or an area closing). Fig. 8.C shows an example of closing by reconstruction of a dilation with a structuring element of size 5*5. The effect of this closing is to fill the small holes inside the connected components. As a result, the distance function of $\varphi^{rec}(X)$ (Fig. 8.D) is much simpler and involves in particular a reduced number regional maxima. The segmentation resulting from this distance function $Dist_{\varphi^{rec}(X)}$ corresponds to a more natural decomposition of the connected components.

The filtering results obtained following this approach are illustrated in Fig. 9. The image on the left side gives an example of simplification with an area open-close filter when the segmentation has been done on $Dist_X$ ($l = \infty$), whereas the image on the right side shows a similar processing but the segmentation has been done on $Dist_{\varphi^{rec}(X)}$. In this last example $l = \infty$, therefore the segmentation is also driven by

A) Connectivity defined by the segmentation of $Dist_X$, ($l = \infty$)

B) Connectivity defined by the segmentation of $Dist_{\varphi^{rec}(X)}$, ($l = \infty$)

Fig. 9. Comparison between the pseudo-connectivity and its robust version (area open-close, $\lambda = 100$)

the ultimate erosion. However, the simplified image does not involve false contours as previously. We are still dealing with a pseudo-connectivity but its definition is more robust.

5. Conclusions

In this paper, two lines of generalization of connected operators have been presented and discussed. The first generalization deals with the simplification criterion. A general scheme relying on binary connected operators can be used to create a large number of new operators. *Complexity* and *Motion* criteria have been presented and discussed. The operator resulting from the first criterion allows an efficient separation of simple objects from complex objects. One can imagine a large number of applications for this operator, one of the most interesting one being segmentation-based representation and coding of images. Indeed, for this kind of application, it is important to identify the objects that can be indeed efficiently represented by a contour-texture approach. The second operator extracts or eliminates objects depending on their motion. Here also a large number of applications can be foreseen, in particular motion estimation and segmentation.

The second generalization concerns the connectivity. We have shown how to modify the notion of connectivity to make it either more or less strict than the usual one. The interest of having a more strict notion has been illustrated. It allows in particular to solve the "leakage" problem of usual connected operators. However, it was shown that this generalization leads only to a pseudo-connectivity. This theoretical problem does not prevent the creation of useful operators, but is a drawback if very strict ($l \gg 1$) connectivities are of interest. If very strict connectivities are necessary, we have shown how to define them with more robustness.

In the future, we will focus on the investigation of the class of processing tools that can be generated from these new connected operators: alternated operators, alternating sequential operators, pyramids, etc.

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