GENERALIZED CONNECTED OPERATORS

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ABSTRACT

This paper[1] deals with the notion of *connected operators*. These operators are becoming popular in image processing because they have the fundamental property of simplifying the signal while preserving the contour information. In a first step, we recall the basic notions involved in binary and gray level connected operators. Then, we show how one can extend and generalize these operators. We focus on two important issues: the connectivity and the simplification criterion. We will show in particular how to create connectivities that are either more or less strict than the usual ones and how to build new criteria.

KEYWORDS: Mathematical morphology, Connected operators, Nonlinear filtering, Complexity criterion, Connectivity, Watersheds, Opening, Closing.

1 INTRODUCTION

The first *connected operators* reported in the literature are known as *opening by reconstruction*. They appeared experimentally[3] for binary images. Initially, they consisted in eroding a binary image by a connected structuring element and in reconstructing all connected components that had not been totally removed by the erosion. It was called *opening* because it is an increasing, anti-extensive and idempotent process. It therefore possesses the three fundamental properties of an algebraic opening. Moreover, it was called *by reconstruction* because it involves a reconstruction process of the connected components that have not been totally removed by the erosion. On this very simple example, one can see that the *binary opening by reconstruction* has the fundamental property of simplifying the signal while preserving the contour information. Indeed, the connected components of the binary image are either totally eliminated (the simplification effect) or perfectly preserved (the contour preservation).

This original idea was fruitful because it led to geodesic operators on sets[4] , to markers for numerical functions[6] , to multi-resolution decomposition with filters by reconstruction[7] , to the concept of dynamics[1] , to area opening[11] . Moreover, an intensive work has been done on the efficient implementation of these transformations[12] . These transformations *by reconstruction* involved not only openings but also closings, alternated filters or even alternating sequential filters. They are becoming very popular because, on experimental bases, they have

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been claimed to simplify the image while preserving contours. This rather surprising property makes them very attractive for a very large number of applications such as noise cancellation, segmentation, pattern recognition, etc.

The study reported in[10,8] revealed that filters by reconstruction belong to a larger class of transformation called *connected operators* that have the fundamental property of interacting with the signal by means of connected components (in the case of binary images) or of *flat zones* (in the case of gray level images). This viewpoint opens the door to various generalization of connected filters. The objective of this paper is to discuss two lines of generalization. The first one deals with the simplification criterion and we will propose a new criterion called *complexity* useful for several applications. The second generalization deals with the connectivity. This notion is closely related to the definition of elementary objects in the scene. The connectivity can be modified to eliminate some drawbacks of connected filters for certain types of images or applications. We will discuss connectivities that are either more or less strict than the "usual" connectivity used in image processing.

The organization of this paper is as follows: section 2 is devoted to the notion of binary and gray level connected operators. Section 3 deals with the *complexity* criterion, whereas section 4 focuses on the notion of connectivity. Finally, section 5 presents the conclusions and discusses possible extensions of this work.

2 BINARY AND GRAY-LEVEL CONNECTED OPERATORS

The original idea of *binary connected operators* relies on the separation of an analysis step and of a decision step as illustrated in Fig. 1. The first one assesses a characteristic of a binary connected component following a given criterion, whereas the second one states whether or not a connected component has to be preserved.

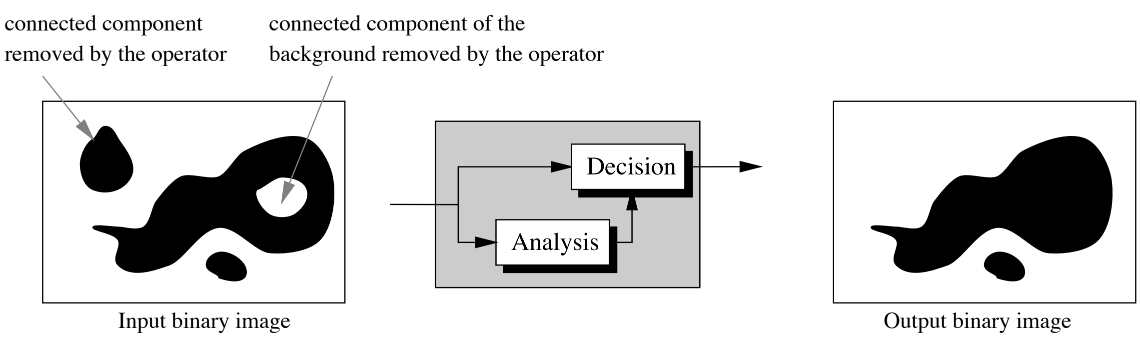


Figure 1: Example of binary connected operator

In[10,8] , the concept of binary connected operators is formally defined as follows: first, a connectivity has to be defined. In the discrete case, this step generally reduces to the definition of a local neighborhood system describing the connections between adjacent pixels. The classical choices involve 4, 6 or 8 connectivity, however, in section 4, we will come back to this notion. Once the connectivity has been selected, the notion of connected operators can be defined as follows:

Definition 2.1 (Binary connected operators). *A binary operator* ψ *is connected when for any binary image* X*, the symmetrical difference* $X \setminus \psi(X)$ *is exclusively composed of connected components of* X *or of its complement* X^c*.*

This is exactly the case of the *binary opening by reconstruction* which acts only by preserving or removing connected components. The extension of the notion of binary connected operators to gray level connected operators relies on the concept of partition[10,8] . Note that, the extension cannot be done directly because the connectivity has no equivalent in the case of gray level functions. Let us recall that a partition of the space E is a set of connected components $\{A_i\}$ which are disjoint and the union of which is the entire space. Each A_i is called a partition class. Moreover, a partition $\{A_i\}$ is said to be *finer* than another partition $\{B_i\}$ if any pair of points belonging to the same class A_i also belongs to a unique partition class B_j. Consider now a binary image and define its *associated partition* as the partition made of the connected components of the binary sets and of their complements. The definition of connected operators can be expressed with associated partitions:

THEOREM 2.2 (BINARY CONNECTED OPERATORS VIA PARTITION). *A binary operator ψ is connected if and only if, for any binary image X, the associated partition of $\psi(X)$ is less fine than the associated partition of X.*

The concept of gray level connected operators can be introduced if we define a partition associated to a function. To this end, the use of *flat zones* was proposed in[10,8] . The set of flat zones of a gray level function f is the set of the largest connected components of the space where f is constant (note that a flat zone can be reduced to a single point). It can be demonstrated[10] that the set of flat zones of a function constitutes a partition of the space. This partition is called the *partition of flat zones* and leads to the following definition:

DEFINITION 2.3 (GRAY LEVEL CONNECTED OPERATORS). *An operator Ψ acting on gray level functions is connected if, for any function f, the partition of flat zones of $\Psi(f)$ is less fine than the partition of flat zones of f.*

There are several ways of creating gray level connected operators. The simplest one consists in extending a binary connected operator. Indeed, as shown in[5,9,2] , any binary operator can generate a gray level operator by *threshold decomposition* and *stacking*. This procedure is illustrated in Fig. 2. The threshold decomposition generates one binary image X_λ for each possible gray level value λ, that is 2^N binary images if the gray levels are quantized with N bits. Note that each binary image X_λ is associated to a specific gray level λ. Each binary image is processed by a binary connected operator ψ. Finally, the stacking consists in reconstructing a gray level image $g = \Psi(f)$ from the set of binary images $\psi(X_\lambda)$:

$$g = \Psi(f) = \bigvee_\lambda \Big(\bigcap_{\mu<\lambda} \psi(X_\mu)\Big) \tag{1}$$

Note that if the binary connected operator ψ is increasing, the stacking can be simplified:

$$g = \Psi(f) = \bigvee_\lambda (\psi(X_\lambda)) \tag{2}$$

Following this procedure, it can be shown[10,8] that the resulting gray level operator Ψ is a connected operator because the partition of flat zones of f is always finer than the partition of flat zones of $\Psi(f)$.

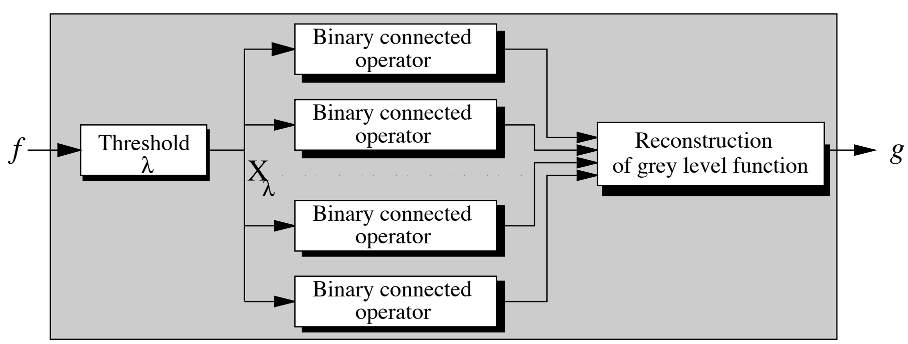


Figure 2: Example of construction of gray level connected operator from a binary connected operator

The processing structure illustrated in Fig. 2 explains why gray level connected operators simplify while preserving the contour information. Indeed, as in the case of binary connected operators, a binary decision process states whether a flat zone has to be preserved or removed. Moreover the decision process is separated from the reconstruction process. Even if the scheme of Fig. 2 is not the only way to create gray level connected operators, in the sequel we will focus on this approach.

This way of creating connected operators opens the door to several generalization. In this paper, we will focus on two points: first, the analysis block of Fig. 1. As can be seen, by modifying the criterion that is assessed in this block, a large set of binary as well as gray level connected operators can be created. Second, the connectivity that is defined during the thresholding operation in Fig. 2. This thresholding operation generally defines the connected components of the binary image to be processed by the binary operator. It therefore defines the elementary image objects on which the decision is going to interact. A modification of the meaning of the connected components after thresholding leads to a different notion of elementary objects.

3 FILTERING CRITERION

3.1 Classical criteria

As examples, let us briefly recall the classical criteria used for the *opening by reconstruction*, the *area opening* and the *h-max* operator. The first two operators can deal with binary (scheme of Fig. 1) as well as gray level images (scheme of Fig. 2) whereas the last one is devoted to gray level images only.

- *Opening by reconstruction*: As discussed in the introduction, this filter preserves all connected components that are not totally removed by a binary erosion by a structuring element of size h. This opening has a size-oriented simplification effect: in the case of gray level images, it removes the bright components that are smaller than the structuring element. By duality, a closing by reconstruction can be defined. Its simplification is similar to that of the opening but on dark components.
- *Gray level area opening*[11]: This filter is similar to the previous one except that it preserves the connected components that have a number of pixels larger than a limit h. It is also an opening which has a size-oriented simplification effect, but the notion of size is different from the one used in the *opening by reconstruction*. By duality an *area closing* can be defined.
- $h - max$ operator: The criterion here is to preserve a connected component of the binary image X_μ if and only if this connected component hits a connected component of the binary image $X_{\mu+h}$. This is an example where the criterion involves two binary images obtained at two different threshold values. The simplification effect of this operator is contrast-oriented in the sense that it eliminates image components with a contrast lower than h. Note that, the $h - max$ is an operator and not a morphological filter because it is not idempotent. By duality, the $h - min$ operator can be defined.

3.2 Complexity criterion

In this section, a new connected operator dealing with the complexity of objects is introduced. The idea is to define a binary connected operator that removes complex binary connected components. To this end, we propose to measure for each connected component the ratio between its perimeter $\mathcal{P}$ and its area $\mathcal{A}$.

$$\text{Complexity criterion} = \mathcal{C} = \mathcal{P}/\mathcal{A} \tag{3}$$

Intuitively, it can be seen that if a connected component has a small area but a very long perimeter, it corresponds to a complex object. For instance in the case of a circle of area $\mathcal{A}$, the complexity is $\mathcal{C} = 2\sqrt{\pi}/\sqrt{\mathcal{A}}$ and for a square of the same area $\mathcal{A}$, then $\mathcal{C} = 4/\sqrt{\mathcal{A}}$. Both objects have the same area, but the circle is more simple than the square.

The complexity criterion is not an increasing criterion because if the set X is included in the set Y, there is *a priori* no relation between their complexity. The reconstruction of the grey level function can therefore be achieved by the formula of Eq. 1. However, in practice, this reconstruction process severely decreases the contrast of the image. This phenomenon is illustrated in Fig. 3 on a 1D signal. One can see the original image, its threshold decomposition and the decision taken by the binary complexity operator on each level. As the complexity criterion is not increasing, the decision taken at a given level does not depend on the decision taken on lower levels. If the reconstruction of the gray level image is done following Eq. 1, the gray level value of the simplified signal corresponds to the level just below the lowest level where the signal has been declared to be "complex". This rule results in a sever loss of contrast. If the reconstruction is defined by Eq. 2, the restitution level is the highest level where the signal was "simple" and the contrast is better preserved. In practice, this second rule leads to more useful results. Finally, note that both reconstruction techniques lead to gray level connected operators.

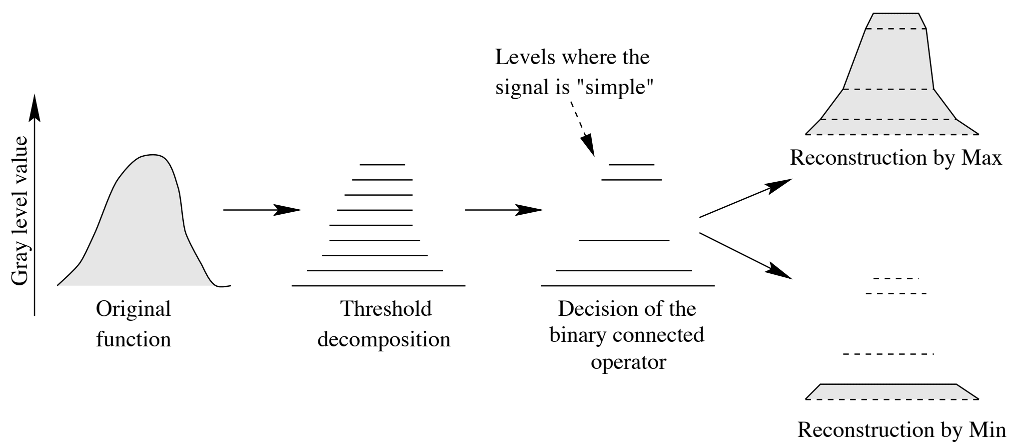


Figure 3: Comparison of reconstruction of the gray level image following equations 1 (by Min) and 2 (by Max)

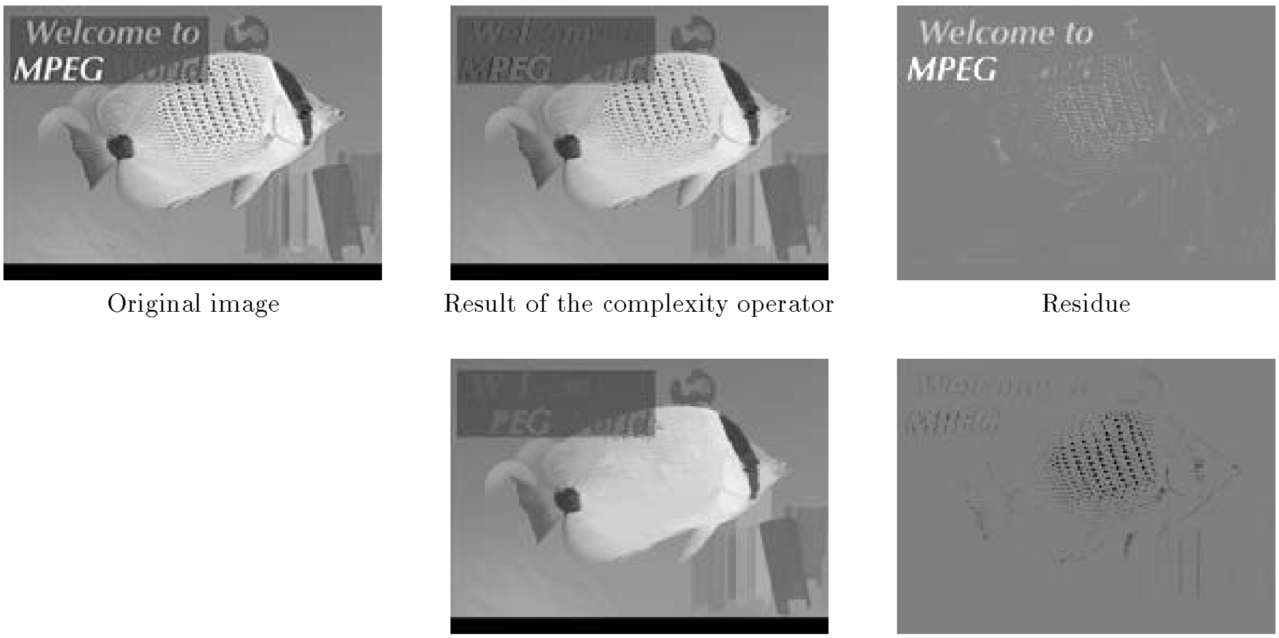
The complexity operator is idempotent, anti-extensive but non increasing. It is therefore not a morphological filters in the strict sense. In practice, this operator removes complex and bright objects from the original image. A dual operator dealing with the complexity of dark objects can be easily defined.

An example of processing can be seen in Fig. 4. The original image is composed of various objects with different complexity. In particular the text and the texture of the fish can be considered as being rather complex by comparison with the shape of the fish and the books that are on the lower right corner. On the right side of Fig. 4, one can see the output of the complexity operator as well as its residue. The residue is the difference between the operator input and output and shows what has been removed by the operator. Finally, on the output of the complexity operator, a dual complexity is applied (second row of Fig. 4). This can be considered as an alternated operator. As illustrated on this example, the complexity operators efficiently separate complex image components (text and texture of fish) while preserving the contours of the objects that have not been removed. In both cases, the filters have removed objects of complexity higher that 1 in the sense of Eq. 3. Note that the simplification effect is not size-oriented, because the filters have removed large objects (the "MPEG" word) as well as small objects (the texture of the fish). The simplification is not contrast-oriented as can be seen by the difference in contrast between "Welcome to" and "MPEG" which have been jointly removed.

Based on the original operator Γ_h and its dual Φ_h, a large set of operators can be created. Let us mention in particular:

- the alternated operator $\Phi_h \Gamma_h$ illustrated in Fig. 4 and its dual $\Gamma_h \Phi_h$,
- the alternating sequential operator: $\Gamma_h \Phi_h \Gamma_{h-1} \Phi_{h-1} \cdots \Gamma_1 \Phi_1$,
- Multiresolution decompositions[7,10,8] .

Complexity operators are useful in a large number of applications where complex objects have to be processed differently from simple objects, involving in particular image analysis task and segmentation-based coding. In this last case, it is very important to be able to select the image components that are more efficiently coded by their contour than by the pixels of its interior. This selection criterion is exactly the complexity criterion that is proposed in this paper.



Original image — Result of the complexity operator — Residue

Result of the dual complexity operator — Residue

Figure 4: Example of processing with the complexity connected operator

4 THE CONNECTIVITY

In the scheme of Fig. 2, the definition of the connectivity is implicitly done in the thresholding block. Indeed, after this first step, the various connected components are assumed to be known and will be processed by the binary connected operators. In discrete space, the notion of connectivity usually relies on the definition of a local neighborhood system that defines the set of pixels that are connected to a given point. In practice, 4-, 6- and 8-connectivity are the most popular choices. In the examples of Fig. 4, a 4-connectivity was used. The objective of this section is to discuss one possible line of extension of the connectivity notion and its influence on the resulting set of connected operators.

4.1 Classical connectivity

The notion of connectivity has been introduced in morphology[9] starting from the following definition:

Definition 4.1 (Connectivity class). *A* connectivity class $\mathcal{C}$ *is defined on the subsets of a set* E *when:*

1. $\emptyset \in \mathcal{C}$ *and* $\forall\ x \in E$, $\{x\} \in \mathcal{C}$
2. *For each family* $\{\mathcal{C}_i\}$ *of* $\mathcal{C}$, $\bigcap \mathcal{C}_i \neq \emptyset \Rightarrow \bigcup \mathcal{C}_i \in \mathcal{C}$

It was shown in[9] that this definition is equivalent to the definition of a family of *connected pointwise openings* $\{\gamma_x, x \in E\}$ associated to each point of E:

Theorem 4.2 (Connectivity characterized by openings). *The definition of a connectivity class* $\mathcal{C}$ *is equivalent to the definition of a family of openings* $\{\gamma_x, x \in E\}$ *such that:*

1. $\forall\ x \in E$, $\gamma_x(\{x\}) = \{x\}$
2. $\forall\ x,\ y \in E$ *and* $X \subseteq E$, $\gamma_x(X)$ *and* $\gamma_y(X)$ *are either equal or disjoint.*
3. $\forall\ x \in E$ *and* $X \subseteq E$, $x \notin X \Rightarrow \gamma_x(X) = \emptyset$

Intuitively, the opening $\gamma_x(X)$ is the connected component of X that contains x. Based on this definition of the connectivity, a generalization was proposed in[9] . It relies on the definition of a new connected pointwise opening

$$\nu_x(X) = \gamma_x(\delta(X)) \bigcap X \text{ , if } x \in X \qquad \text{and} \qquad \nu_x(X) = \emptyset \text{ , if } x \notin X \tag{4}$$

where δ is an extensive dilation. It can be demonstrated that this new function is indeed a connected pointwise opening and therefore defines a new connectivity. This connectivity is less "strict" than the usual ones in the sense that it considers that two objects that are close to each other (that is they touch each other if they are dilated by δ) belong to the same connected component. This generalization can lead to interesting new connected filters, however, in order to have a flexible tool one would like also to define connectivities that are more "strict" that the usual ones, that is they should split what is usually considered as one connected component. The purpose of the following section is to define such a tool and to present it in a framework where one can progressively go from strict connectivity to loose connectivity.

4.2 From loose connectivity to strict pseudo-connectivity

Let us describe the intuitive idea of our generalization on the example of Fig. 5. This figure shows a binary image resulting from the thresholding of the image of Fig. 4 and its 4-connected components. There are 42 connected components. As can be seen, the main part of the fish is merged with its head and with some part of the books below it. Intuitively, one would like to segment this connected component and process separately each of its main parts.

The definition of a "strict" connectivity leads to the segmentation of the binary connected components. This segmentation can naturally be done by classical binary segmentation tools (see[6] and the references herein). One of the simplest approaches consists in computing the distance function $Dist_X$ on the binary set X and in computing the *watershed* of $-Dist_X$. The *watershed* transform associates to each minima of $-Dist_X$ a region called a *catchment basin*. Note that the minima of $-Dist_X$ are the maxima of the distance function, in other words, they

Figure 5: Binary image (left) and its 4-connected components (right)

correspond to the *ultimate erosions* of the set. This approach is illustrated on Fig. 6 and results in the creation of 48 connected components. As can be seen the connected component corresponding to the fish has been segmented in five different connected components.

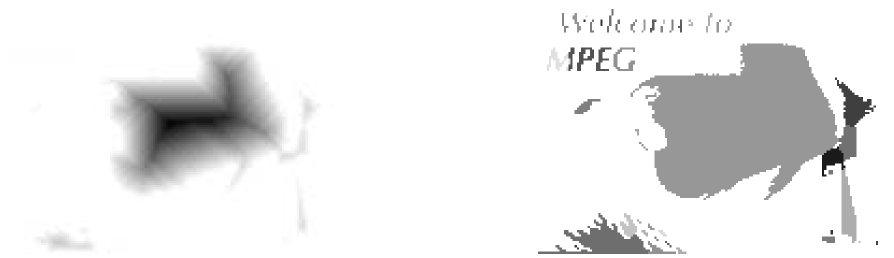


Figure 6: Distance function $-Dist_X$ (left) and the resulting connected components (right)

If this segmentation driven by the ultimate erosion creates too many connected components, the number of connected components can be defined by the number of connected components in the classical sense of an erosion of size l of X. This can be implemented via the segmentation of a thresholded version of the distance function:

$$\mathcal{D}_{X,l} = -(Dist_X \bigwedge l) \tag{5}$$

An example can be seen in Fig. 7. The distance function has been thresholded at 3 and produces 46 connected components. Now the fish has been segmented in three components. The parameter l of $\mathcal{D}_{X,l}$ allows to go progressively from the classical connectivity when $l = 0$ to the extreme case where the number of connected components are defined by the number of ultimate erosions when $l = \infty$.

Note that the approach can be easily modified to include within the same framework the "loose" connectivity described by the connected pointwise opening ν_x of Eq. 4. Indeed, consider the distance function of the background $Dist_{X^C}$ and create the function $D_X = Dist_X - Dist_{X^C}$ (see Fig. 8). This function defines all the possible erosions and dilations with the structuring element used to define the distance. The segmentation by watershed of the thresholded version of D_X defined as in Eq. 4 by:

$$\mathcal{D}_{X,l} = -(D_X \bigwedge l) \tag{6}$$

leads to a definition of the connected components of X which is:

Figure 7: Thresholded distance function $-\mathcal{D}_{X,3}$ (left) and the resulting connected components (right)

- equal to the classical connectivity $\mathcal{C}$ if $l = 0$,
- looser than $\mathcal{C}$ if $l < 0$: two components are connected if their distance is smaller that l,
- more strict than $\mathcal{C}$ if $l > 0$: one component is segmented by the watershed algorithm in a number of connected components equals to the number of connected components (in the sense of $\mathcal{C}$) of its erosion of size l.

Distance function of X: $Dist_X$

Distance function of the background: $Dist_{X^C}$

Global distance function: D_X

Figure 8: Distance functions

Let us define $\mathcal{CB}^l_x(X)$ the transformation that assigns to x the *catchment basin* of the function $\mathcal{D}_{X,l}$ that contains x. Consider now the operator:

$$\mathcal{CC}^l_x(X) = \mathcal{CB}^l_x(X) \bigcap X, \text{ if } x \in X \qquad \text{and} \qquad \mathcal{CC}^l_x(X) = \emptyset, \text{ if } x \notin X \tag{7}$$

This transformation reduces to the classical connected pointwise opening γ_x when $l = 0$. Moreover, if $l < 0$, it is equal to the connected pointwise opening ν_x defined by Eq. 4. Therefore, for $l \leq 0$, the operator $\mathcal{CC}^l_x$ defines a real connectivity. This is however *not* the case for $l > 0$. Indeed, in that case, all conditions of theorem 4.2 are met except one: $\mathcal{CC}^l_x$ is not increasing and therefore not an opening. A counterexample can be seen in Fig. 9. For $l > 0$, The operator $\mathcal{CC}^l_x$ almost defines a connectivity that will be called a *pseudo-connectivity* in the following. This is a drawback, but, using the watershed as segmentation tool, our main concern is to segment the component of X in a small number of regions and to keep as much as possible the contour information of X, because it is one of the main attractive properties of connected operators. Moreover, it can be shown that the regions of the space where $\mathcal{CC}^l_x$ is actually not increasing increases with the value of l. For small values of l, this theoretical problem does not prevent the creation of useful operators.

Fig. 10 illustrates several examples of area opening[11] with several notions of connectivity. The classical area opening can be seen in Fig. 10.c and .f for two different values of the area parameter. In both cases, small

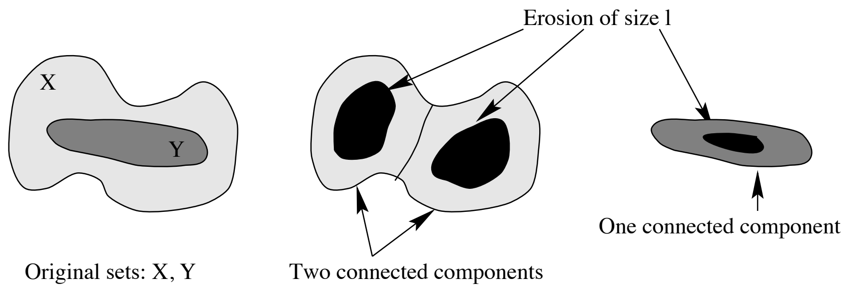


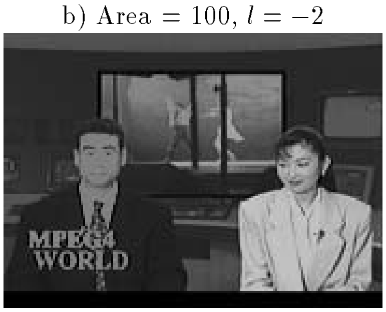
Figure 9: Counterexample: $Y \subset X$ but for $l > l_0$ the connected components of Y are not included in the connected components of X

a) Original Image

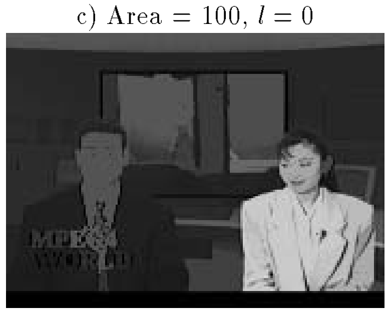
b) Area = 100, $l = -2$

c) Area = 100, $l = 0$

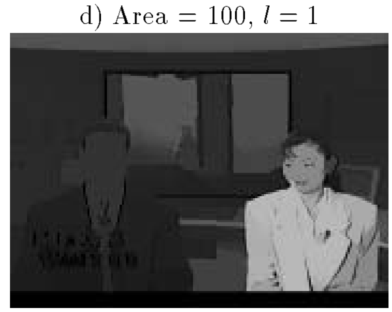
d) Area = 100, $l = 1$



e) Area = 800, $l = -2$



f) Area = 800, $l = 0$



g) Area = 800, $l = 1$

Figure 10: Example of area filtering with loose connectivity and strict pseudo-connectivity

bright objects have been removed. One can see one drawback of these filters: small objects like the letters of the "MPEG" word have been removed and merged with surrounding areas. Moreover, their gray level values depend on the surrounding regions. As can be seen, these objects have not been totally removed from the image because the letters have been connected between themselves and with the shirt of the man. This is the classical problem of "leakage" of connected operators. This problem is solved if the pseudo-connectivity is used. Fig. 10.d and .g present the same area filter but with a threshold value of $l = 1$ on the distance function. In this case, thin connections between components are broken and the final result corresponds more to a "natural" size-oriented simplification. Fig. 10.b and .e show the case where $l = -2$. Here, the effect is to merge connected components if they are close to each others.

The example of Fig. 11 illustrates the problems that appear if a high threshold value on the distance function is used. As can be seen, the contour preservation property is lost for certain objects.

a) Area = 100 b) Area = 800

Figure 11: Area opening with pseudo-connectivity with $l = 7$

Finally, the last example of Fig. 12 shows two examples of simplified alternating sequential operators. With the notations of the end of section 3.2, the operators are defined by: $\Gamma_h \Phi_h \Gamma_{h/2} \Phi_{h/2}$. The figure compares area- and complexity-oriented simplification with the pseudo-connectivity obtained with $l = 1$. As can be seen, one image actually corresponds to large objects of the scene, while the second one only preserves simple objects.

a) Area b) Complexity

Figure 12: Area- and complexity-oriented simplification with pseudo-connectivity with $l = 1$

5 CONCLUSIONS

In this paper, two lines of generalization of connected operators have be presented and discussed. The first generalization deals with the simplification criterion. A general scheme relying on binary connected operators can be used to create a large number of new operators. A complexity criterion has been proposed. The resulting operator allows an efficient separation of simple objects from complex objects. The complexity is an example of non increasing criterion and we have discussed its influence of the construction of the operator. This example can be viewed as a basis for the construction of new filtering criteria in the future.

The second generalization concerns the connectivity. We have shown how to modify the notion of connectivity to make it either more or less strict than the usual one. The interest of having a more strict notion has been illustrated. It allows in particular to solve the "leakage" problem of usual connected operators. However, it was shown that this generalization leads only to a pseudo-connectivity. This theoretical problem does not prevent the creation of useful operators, but is a drawback if very strict ($l \gg 1$) connectivities are of interest. In the future, we will focus on modifications of the approach to deal with very strict pseudo-connectivity.

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